Thompson Sampling : an asymptotically optimal finite-time analysis

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The stochastic MAB with Bernoulli rewards

K indepedent arms.

- $\mu_1, \ldots, \mu_K$ **unknown parameters**
- $(Y_{a,t})_t$ is i.i.d. with distribution $\mathcal{B}(\mu_a)$

The parameter of the best arm is $\mu^* = \max_{a=1\ldots K} \mu_a$

- At time t, the forecaster chooses arm A_t and gets reward $R_t = Y_{A_t,t}$.
- Goal : Design a strategy A_t **minimizing the cumulative regret**:

$$\mathcal{R}(T) := T\mu^* - \mathbb{E}\left[\sum_{t=1}^{T} R_t\right] = \sum_{a \in A} (\mu^* - \mu_a)\mathbb{E}[N_{a,T}]$$

Asymtotically optimal bandit algorithms

- Lai and Robbins' lower bound on the regret of a consistent policy:

$$\mu_a < \mu^* \Rightarrow \liminf_{T\to\infty} \frac{\mathbb{E}[N_{a,T}]}{\ln T} \geq \frac{1}{K(\mu_a, \mu^*)}$$

or equivalently

$$\liminf_{T\to\infty} \frac{\mathbb{E}[\mathcal{R}(T)]}{\ln(T)} \geq \sum_{a:\mu_a<\mu^*} \frac{\mu^* - \mu_a}{K(\mu_a, \mu^*)}$$

with

$$K(p,q) := p \ln \frac{p}{q} + (1-p) \ln \frac{1-p}{1-q}.$$

- A bandit algorithm is **asymptotically optimal** if

$$\mu_a < \mu^* \Rightarrow \limsup_{T\to\infty} \frac{\mathbb{E}[N_{a,T}]}{\ln T} \leq \frac{1}{K(\mu_a, \mu^*)}$$

Some sucessfull frequentist algorithms

A family of **optimistic index policies** based on an **upper confidence bound** for the empirical mean of the rewards:

- UCB [Auer et al. 02] and variants:

$$\mathbb{E}[N_{a,T}] \leq \frac{K_1}{2(\mu_a - \mu^*)^2} \ln T + K_2, \quad \text{with } K_1 > 1.$$

- KL-UCB [Cappé, Garivier, Maillard, Stoltz, Munos 11] uses the index:

$$u_{a,t} = \underset{x > \frac{S_{a,t}}{N_{a,t}}}{\text{argmax}} \left\{ K\left(\frac{S_{a,t}}{N_{a,t}}, x\right) \leq \frac{\ln(t) + c \ln\ln(t)}{N_{a,t}} \right\}$$

For all $\epsilon > 0$, there exists a constant K_ϵ such that:

$$\mathbb{E}[N_{a,T}] \leq \frac{1+\epsilon}{K(\mu_a, \mu^*)} \ln T + K_\epsilon$$

A Bayesian view on the MAB

Imagine we are given independent priors on the parameters of each arm:

- $\mu_a \overset{i.i.d.}{\sim} \mathcal{U}([0,1])$
- $(Y_{a,t})_t$ is i.i.d. conditionally to μ_a with distribution $\mathcal{B}(\mu_a)$
- The posterior on arm a at time t is

$$\pi_{a,t} = \text{Beta}\left(S_{a,t}+1, N_{a,t}-S_{a,t}+1\right).$$

Bayesian algorithms uses this posterior $\pi_{a,t}$ to choose A_t.

$\Rightarrow$ We still focus on frequentist guarantees (asymptotic optimality) for Bayesian algorithms

A Bayesian Upper Confidence Bound algorithm

- Bayes-UCB [Kaufmann et al. 12] is the index policy associated with

$$q_{a,t} := Q\left(1 - \frac{1}{t \ln(t)^c}, \pi_{a,t}\right)$$

This Bayesian algorithm is asymptotically optimal

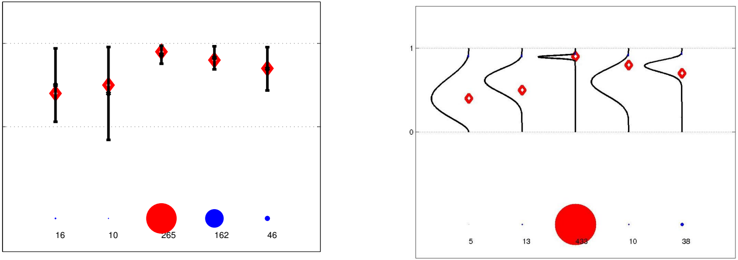

Figure: UCB versus Bayes-UCB

Thompson Sampling : a new kind of optimism?

- A very simple algorithm:

$$\forall a \in \{1..K\}, \quad \theta_{a,t} \sim \pi_{a,t}$$
$$A_t = \text{argmax}_a \ \theta_{a,t}$$

- Recent interest for this algorithm:
  - partial analysis proposed
    [Granmo 2010][May, Korda, Lee, Leslie 2011]
  - extensive numerical study beyond the Bernoulli case
    [Chapelle, Li 2011]
  - first logarithmic upper bound on the regret
    [Agrawal,Goyal 2012]

An optimal regret bound for Thompson Sampling

Assume the first arm is the unique optimal and $\Delta_a = \mu_1 - \mu_a$.

- Known result : [Agrawal,Goyal, 2012]

$$\mathbb{E}[\mathcal{R}(T)] \leq C \left(\sum_{a=2}^{K} \frac{1}{\Delta_a} \right) \ln(T) + o_{\mu}(\ln(T))$$

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- Our improvement :

Theorem 2 $\forall \epsilon > 0$,

$$\mathbb{E}[\mathcal{R}(T)] \leq (1 + \epsilon) \left(\sum_{a=2}^{K} \frac{\Delta_a}{K(\mu_a, \mu^*)} \right) \ln(T) + o_{\mu,\epsilon}(\ln(T))$$

Step 1: Decomposition

- We adapt an analysis working for optimistic index policies:

$$A_t = \text{argmax}_a l_{a,t}$$

$$\mathbb{E}[N_{a,T}] \leq \underbrace{\sum_{t=1}^{T} \mathbb{P}\left(l_{1,t} < \mu_1\right)}_{o(\ln(T))} + \underbrace{\sum_{t=1}^{T} \mathbb{P}\left(l_{a,t} \geq l_{1,t} > \mu_1, A_t = a\right)}_{\ln(T)/K(\mu_a,\mu_1)+o(\ln(T))}$$

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$\Rightarrow$ Does NOT work for Thompson Sampling

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- Our decomposition for Thompson Sampling is

$$\mathbb{E}[N_{a,T}] \leq \sum_{t=1}^{T} \mathbb{P}\left(\theta_{1,t} \leq \mu_1 - \sqrt{\frac{6 \ln t}{N_{1,t}}}\right) + \underbrace{\sum_{t=1}^{T} \mathbb{P}\left(\theta_{a,t} > \mu_1 - \sqrt{\frac{6 \ln t}{N_{1,t}}}, A_t = a\right)}_{(*)}$$

Step 2: Linking quantiles to other known indices

- We introduce the following quantile:

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- We know from previous work [Kaufmann et al.] that

$$q_{a,t} < u_{a,t}$$

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- Introducing the quantile $q_{a,t}$:

$$\sum_{t=1}^{T} \mathbb{P}\left(\theta_{a,t} > \mu_1 - \sqrt{\frac{6 \ln t}{N_{1,t}}}, A_t = a\right)$$

$$\leq \sum_{t=1}^{T} \mathbb{P}\left(q_{a,t} > \mu_1 - \sqrt{\frac{6 \ln t}{N_{1,t}}}, A_t = a\right) + \underbrace{\sum_{t=1}^{T} \mathbb{P}\left(\theta_{a,t} > q_{a,t}\right)}_{\leq 2}$$

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- Then the KL-UCB index $u_{a,t}$:

$$\sum_{t=1}^{T} \mathbb{P}\left(\theta_{a,t} > \mu_1 - \sqrt{\frac{6\ln t}{N_{1,t}}}, A_t = a\right)$$
$$\leq \sum_{t=1}^{T} \mathbb{P}\left(u_{a,t} > \mu_1 - \sqrt{\frac{6\ln t}{N_{1,t}}}, A_t = a\right) + 2$$

Final decomposition

- The final decomposition is:

$$\mathbb{E}[N_{a,t}] \leq \underbrace{\sum_{t=1}^{T} \mathbb{P}\left(\theta_{1,t} \leq \mu_1 - \sqrt{\frac{6\ln t}{N_{1,t}}}\right)}_{A}$$

$$+ \underbrace{\sum_{t=1}^{T} \mathbb{P}\left(u_{a,t} > \mu_1 - \sqrt{\frac{6\ln t}{N_{1,t}}}, A_t = a\right)}_{B} + 2$$

Step 3: One extra ingredient for bounding term A and B

- We state **a fundamental deviation result** :

Proposition 1 There exists constants $b = b(\mu_1, \mu_2) \in (0, 1)$ and $C_b < \infty$ such that:

$$\sum_{t=1}^{\infty} \mathbb{P}\left(N_{1,t} \leq t^b\right) \leq C_b.$$

Understanding the deviation result

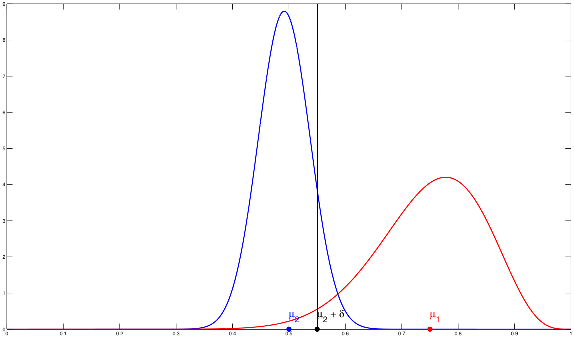
- Recall the result

There exists constants $b = b(\mu_1, \mu_2) \in (0, 1)$ and $C_b < \infty$ such that

$$\sum_{t=1}^{\infty} \mathbb{P}\left(N_{1,t} \leq t^b\right) \leq C_b.$$

- Where does it come from?

$$\left\{N_{1,t} \leq t^b\right\} = \{\text{there exists a time range of length at least } t^{1-b} - 1 \text{ with no draw of arm } 1\}$$



Assume that :

- on $\mathcal{I}_j = [\tau_j, \tau_j + \lceil t^{1-b} - 1 \rceil]$ there is no draw of arm 1
- there exists $\mathcal{J}_j \subset \mathcal{I}_j$ such that $\forall s \in \mathcal{J}_j, \forall a \neq 1$, $\theta_{a,s} \leq \mu_2 + \delta$

Then :

- $\forall s \in \mathcal{J}_j$, $\theta_{1,s} \leq \mu_2 + \delta$

$\Rightarrow$ This only happens with small probability

Conclusion and perspectives

Thompson Sampling in the Bernoulli setting:

- has the same theoretical guarantees than known optimal algorithms (KL-UCB, Bayes-UCB)
- and displays excellent empirical performance

The proof we give :

- is close to the analysis of optimistic bandit algorithms
- also gives a deviation result on the number of draws of optimal arms

Can Thompson Sampling be extended to more general settings?

- Contextual bandit ([Agrawal, Goyal, Thompson Sampling for Contextual Bandits with Linear Payoffs, sept 2012])
- Model-based Bayesian reinforcement learning

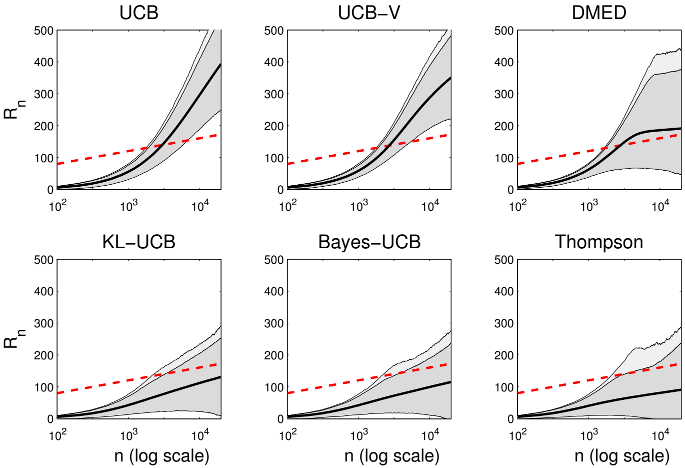


Figure: Regret as a function of time (on a log scale) for a 10 arms problem

Thompson Sampling outperforms other optimal algorithms

Any question ?