TD4 - (Likelihood Ratio) Testing

Exercise 1  We collect one sample $X$ from a Poisson distribution with parameter $\lambda$. We recall that its probability mass function is given by

$$\forall k \in \mathbb{N}, \ f_\lambda(x) = \frac{\lambda^k}{k!} e^{-\lambda}.$$  

We want to test $H_0 : (\lambda = 5)$ against $H_1 : (\lambda = 10)$ at level $\alpha = 0.05$, based on $X$.

1. Prove that a randomized Neyman-Pearson test can be formulated as

$$\tilde{D}(X) = 1, \quad \text{if } X > t$$
$$\tilde{D}(X) = \gamma, \quad \text{if } X = t$$
$$\tilde{D}(X) = 0, \quad \text{if } X < t.$$  

2. Using that $P_{Z \sim P(5)} (Z > 9) = 0.032$ and $P_{Z \sim P(5)} (Z > 8) = 0.068$, deduce that $t = 9$ and $\gamma = 1/2$.

3. What is the power of this test?

Exercise 2  We collect iid data $X_1, \ldots, X_n$ from an exponential distribution with parameter $\theta$. We recall that its density is given by

$$\forall x \in \mathbb{R}, \ f_\theta(x) = \theta \exp(-\theta x) 1_{[0, +\infty)}(x).$$

1. Propose a Uniformly More Powerful test of level $\alpha$ for the test

$$H_0 : (\theta \leq \theta_0) \quad \text{against} \quad H_1 : (\theta > \theta_0).$$

2. Can we propose a UMP($\alpha$) test for $H_0 : (\theta = \theta_0)$ against $H_1 : (\theta \neq \theta_0)$?

Exercise 3  We consider a two-sample testing problem in which we observe $X_1, \ldots, X_{n_1} \sim \mathcal{N}(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_{n_2} \sim \mathcal{N}(\mu_2, \sigma^2)$ where $(\mu_1, \mu_2) \in \mathbb{R}^2$ and want to test

$$H_0 : (\mu_1 = \mu_2) \quad \text{against} \quad H_1 : (\mu_1 \neq \mu_2).$$

We denote by $Z = (X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$ the observation.

1. Assuming the variance $\sigma^2$ is known, compute

$$\log \tilde{\Lambda}(Z) = \log \frac{\sup_{\mu_1, \mu_2} L(Z; \mu_1, \mu_2)}{\sup_{\mu_1 = \mu_2} L(Z; \mu_1, \mu_2)}.$$  

2. Propose a (non-asymptotic) LRT test of level $\alpha$. Compute its power function.

3. Compare it with the asymptotic LRT test provided by Wilk’s theorem.
Exercise 4  We consider a so-called one-parameter canonical exponential family, in which the density wrt to some reference measure is

\[ f_\theta(x) = h(x) \exp(\theta x - b(\theta)) \]

where \( b \) is some twice differentiable function that is furthermore strictly convex \((b'' > 0)\). We admit that these assumption are sufficient to be in a regular model. We denote by \( \mu(\theta) = \mathbb{E}_\theta[X] \) the expectation of the distribution parameterized by \( \theta \).

1. Prove that \( b'(\theta) = \mathbb{E}_\theta[X] \) and \( b''(\theta) = \text{Var}_\theta[X] \).
2. Deduce that the mapping \( \theta \mapsto \mu(\theta) \) is one-to-one. We denote by \( \mu^{-1} \) its inverse.
3. Compute \( \hat{\theta}_n \), the maximum likelihood estimator of \( \theta \).
4. We introduce \( K(\theta, \theta') \), the Kullback-Leibler divergence between \( P_\theta \) and \( P_{\theta'} \), defined as

\[ K(\theta, \theta') = \mathbb{E}_\theta \left[ \log \frac{f_\theta(X)}{f_{\theta'}(X)} \right] \]

Prove that

\[ K(\theta, \theta') = (\theta - \theta')\mu(\theta) - b(\theta) + b(\theta') \]

5. Deduce the following inequality: for all \( \theta \in \Theta \),

\[ \log \frac{L(X_1, \ldots, X_n; \hat{\theta}_n)}{L(X_1, \ldots, X_n; \theta)} = nK(\hat{\theta}_n, \theta) \]

6. Prove that the (generalized) log-likelihood ratio associated to the test

\[ \mathcal{H}_0 : (\theta \leq \theta_0) \text{ against } \mathcal{H}_1 : (\theta > \theta_0) \]

satisfies

\[ \log \Lambda(X) = nK(\hat{\theta}_n, \theta_0) 1(\hat{\theta}_n \geq \theta_0). \]

7. We admit the following concentration inequality (called a Chernoff inequality):

\[ \forall \theta \in \Theta, \forall x > 0, \quad \mathbb{P}_\theta(\hat{\theta}_n > \theta, nK(\hat{\theta}_n, \theta) > x) \leq e^{-x} . \]

Propose a LRT of level \( \alpha \). Is this test UMP(\( \alpha \))?
8. Propose a test of level \( \alpha \) for testing

\[ \mathcal{H}_0 : (\mu = \mu_0) \text{ against } \mathcal{H}_1 : (\mu \neq \mu_0) . \]

Compare it to the asymptotic test of level \( \alpha \) obtained using Wilk’s theorem, for \( \alpha = 0.05 \). Which one will have the largest power?
Exercise 5  The scientist Mendel (considered as the father of genetics) did the following experiment. He bred two different kind of peas: one with round yellow seeds and one with wrinkled green seeds. There are four types of progeny: round yellow (1), wrinkled yellow (2), round green (3) and wrinkled green (4). For each individual in the progeny, we denote by $p_i$ the probability that it is of type $i$.

Assuming that the individual are independent when there are $n$ individuals, the number of individual of each kind $N_i = (N_{n,1}, N_{n,2}, N_{n,3}, N_{n,4})$ follows a so-called multinomial distribution with parameter $n$ and $p = (p_1, p_2, p_3, p_4)$, for which

$$P(N_n = (n_1, n_2, n_3, n_4)) = \frac{n!}{n_1!n_2!n_3!n_4!} \prod_{i=1}^4 p_i^{n_i}.$$  

Mendel’s inheritance theory predicts that $p$ is equal to

$$p_0 = \left(\frac{9}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16}\right).$$

He did $n = 556$ experiments and observed $N_n = (315, 101, 108, 32)$. We want to test

$$H_0 : (p = p_0) \text{ against } H_1 : (p \neq p_0).$$

1. Perform a Likelihood Ratio Test of this hypothesis. Does it reject $H_0$?

2. For the above testing problem with multinomial data it is common to use another test, called Pearson’s $\chi^2$ test. When there are $k$ possible types, this test is based on the test statistic $T_n = \sum_{i=1}^k \frac{(N_{n,i} - np_{0,i})^2}{np_{0,i}}$

which is proved to satisfy $T_n \sim \chi^2_{k-1}$. Perform a $\chi^2$ test. Does it reject the hypothesis?

3. Do you think that using statistical testing is appropriate to validate a theory?

Exercise 6  Let $Z$ denote a random variable with density

$$x \mapsto \frac{1}{\lambda} \exp\left(-\frac{x - \theta}{\lambda}\right) \cdot 1_{[\theta, +\infty)}(x),$$

where $\lambda > 0$ and $\theta \in \mathbb{R}$ are unknown. Let $(X_1, \ldots, X_n)$ be an $n$-sample of i.i.d. variables with the same distribution as that of $Z$.

1. What is the MLE $(\widehat{\lambda}_n, \widehat{\theta}_n)$ of the unknown parameter $(\lambda, \theta)$? Evaluate the bias of this estimator and provide an unbiased estimator of $(\lambda, \theta)$.

2. Assume that one is to test $H_0 : (\lambda = 1)$ against $H_1 : (\lambda \neq 1)$. Prove that the (Generalized) Likelihood Ratio Test rejects the null hypothesis $\lambda = 1$ whenever $\widehat{\lambda}_n \notin [a, b]$, where $a$ and $b$ satisfy some equations to be clarified.

3. Let us now assume that $\lambda$ is known, and that $\alpha \in ]0, 1[$ is given. Use the MLE of $\theta$ to build a confidence interval with confidence level $1 - \alpha$ for the parameter $\theta$. Deduce a statistical test of $H_0 : (\theta = \theta_0)$ against $H_1 : (\theta \neq \theta_0)$.

4. Let us finally assume that $\theta$ is known and $\widehat{\lambda}_n$ denotes the MLE of $\lambda$. By using the fact that the exponential distribution with parameter $1/2$, i.e., $\mathcal{E}(1/2)$, is a $\chi^2$ distribution with 2 degrees of freedom, what is the distribution of $2n\widehat{\lambda}_n/\lambda$? Deduce a confidence interval for $\lambda$ with confidence level $1 - \alpha = 0.95$ when $n = 13$. 