Sequential Decision Making

Lecture 4 : Markov Decision Processes

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What is Reinforcement Learning ?

- ➜ learning by "trial and error"
- ➜ learning to behave in an unknown, shochastic environement by maximizing some real-valued reward signal

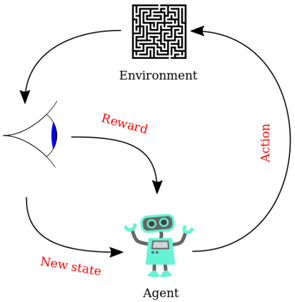
Example : learning to bike without a perfect knowledge of physics

Key RL concepts

A learning agent *sequentially* interacts with its environment by performing actions. Each action

- provides an instantaneous reward
- leads to an evolution of the agent's state

Agent's goal : act so as to maximize its total reward



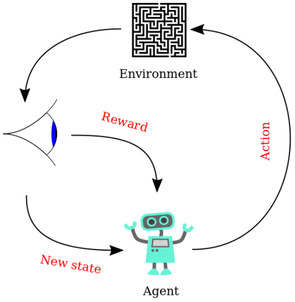
source : Wikipedia

Key RL concepts

Keywords (high-level) :

- Reward : instantaneous feedback received after acting
- Policy : strategy to choose an action in a given state
- Value : total reward the agent can get in some state by following some policy

Agent's goal : find a policy that maximizes the value in each state



source : Wikipedia

RL successes : Games (1/2)

From Backgammon...

1992, TD-gammon

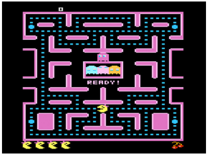
... to Go

2015, AlphaGo
2017, AlphaGo Zero

➜ RL agents learn new types of strategies

RL successes : Games (2/2)

- Learning to play from pixels (and rewards) : Atari Games
  2010+ Deep Reinforcement Learning



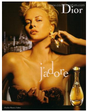
- Recent challenges : multi-player / partial information games

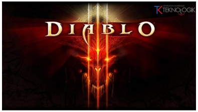
OpenAI Five (2019)

Pluribus (2019)

RL sucessess : Content Optimization

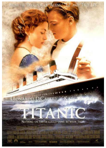
- online advertisement

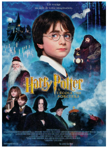


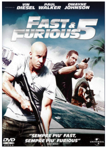


➜ action : display an add / reward : click

- (sequential) recommender systems



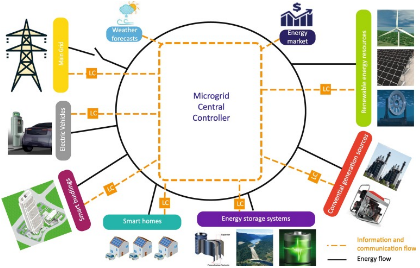




➜ action : recommend a movie / reward : rating

RL : Many potential applications

- Smart grid / microgrid management



source : ScienceDirect.com

Actions :

- charge or discharge storage systems
- turn on or off renewable energy source
- buy energy from the market

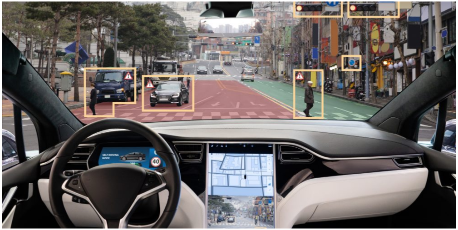
...

Reward : - Cost

RL : Many potential applications

- Autonomous robotics

- Self-driving cars ?



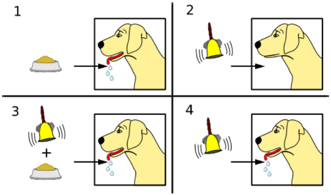
History of RL

- Learning to behave from rewards : an old idea from psychology

- 1900s : observation of animal behavior
  (e.g. Thorndike 1911 "Law of Effect")

Of several responses made to the same situation, those which are accompanied or closely followed by satisfaction to the animal will [...] be more likely to recur.

- 1920s : Pavlov work on conditionnal reflexes
  first occurence of "reinforcement" in animal learning



source : Wikipedia

History of RL

• Learning to behave from rewards : an inspiration from the brain ?

- 1950s : first experiments on electric brain stimuli for controlling mice behavior (Oak and Miller 1954)

➜ hypothesis that dopamine broadcast rewards signal to the brain

History of RL

- Some steps towards computational RL

- 1950s, Shannon's machines : "Theseus", a mice finding how to get out of a maze, a chess player, a Rubik's cube solver
- 1957, Bellmann : Dynamic Programming (control of dynamical systems)
- 1961, Minsky "Towards artificial intelligence"
- 1978, Sutton : Temporal Difference Learning (artificial intelligence)
- 1989, Watkins : Q-Learning algorithm

Nowadays, reinforcement learning is mostly formalized as learning an optimal policy in an incompletely-known Markov Decision Process.

Markov Decision Process

A Markov Decision Process (MDP) models a situation in which repeated decisions (= choices of actions) are made. MDP provides models for the consequence of each decisions :

- in terms of reward
- in terms of the evolution of the system's state

In each (discrete) decision time $t = 1, 2, \ldots$, a learning agent

- selects an action a_t based on his current state s_t (or possibly all the previous observations),
- gets a reward $r_t \in \mathbb{R}$ depending on his choice,
- transits to a new state s_{t+1} depending on his choice.

Markov Decision Process

A MDP is parameterized by a tuple $(\mathcal{S}, \mathcal{A}, R, P)$ where

- $\mathcal{S}$ is the state space
- $\mathcal{A}$ is the action space
- $R = (\nu_{(s,a)})_{(s,a)\in\mathcal{S}\times\mathcal{A}}$ where $\nu_{(s,a)} \in \Delta(\mathbb{R})$ is the reward distribution for the state-action pair (s, a)
- $P = (p(\cdot|s, a))_{(s,a)\in\mathcal{S}\times\mathcal{A}}$ where $p(\cdot|s, a) \in \Delta(\mathcal{S})$ is the transition kernel associated to the state-action pair (s, a)

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[Bellman 1957, Howard 1960, Blackwell 70s...]

Markov Decision Process

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Goal : (made more precise later) select actions so as to maximize some notion of *expected* cumulated rewards

Mean reward of action a in state s

$$r(s, a) = \mathbb{E}_{R\sim\nu_{(s,a)}}[R]$$

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- **The tabular case** : finite state and action spaces

$$\begin{aligned} \mathcal{S} &= \{1, \dots, S\} \\ \mathcal{A} &= \{1, \dots, A\} \end{aligned}$$

For every $s, s' \in \mathcal{S}$, $a \in \mathcal{A}$, $p(s'|s,a) = \mathbb{P}\left(s_{t+1} = s' | s_t = s, a_t = a\right)$.

Why Markov ?

In an MDP, the sequence of sucessive states / actions / rewards

$$s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}, r_{t-1}, s_t$$

satisfies some extension of the Markov property :

$$\begin{aligned}\mathbb{P}\left(s_t = s, r_{t-1} = r \middle| s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}\right) \\ = \mathbb{P}\left(s_t = s, r_{t-1} = r \middle| s_{t-1}, a_{t-1}\right)\end{aligned}$$

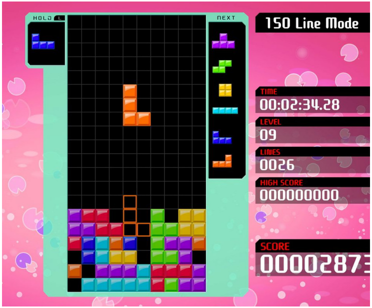
(discrete action and reward)

Definition

A Markov chain on a discrete space $\mathcal{X}$ is a stochastic process $(X_t)_{t\in\mathbb{N}}$ that satisfies the Markov property :

$$\mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}, \dots, X_0 = x_0) = \mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}).$$

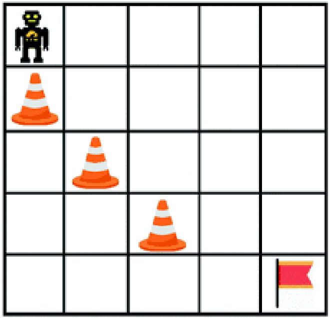
Example : Tetris



- **State** : current board and next blocks to add
- **Action** : orientation + position of the dropped block
- **Reward** : increment in the score/ number of lines
- **Transition** : new board + randomness in the new block

➜ difficulty : large state space !

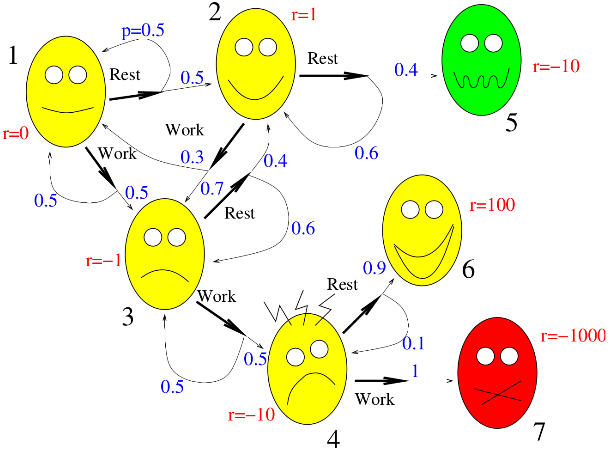
Example : Grid world



- **State** : position of the robot
- **Actions** : $\leftarrow,\uparrow,\rightarrow,\downarrow$
- **Transitions** : (quasi) deterministic
- **Rewards** : depends on the behavior to incentivise (positive or negative rewards on some states / -1 for each step before a goal...)

➜ possible difficulty : sparse rewards

Example : The Student Dilemma



credit : Rémi Munos, Alessandro Lazaric

(running) Example : Retail Store Management

You own a bike store. During week t, the (random) demand is D_t units. On Monday morning you may choose to command a_t additional units : they are delivered immediately before the shop opens.

For each week :

- Maintenance cost : h per unit left in your stock
- Ordering cost : c per unit ordered + fix cost c_0 if an order is placed
- Sales profit : p per unit sold

Constraints :

- your warehouse has a maximal capacity of M bikes (any additional bike gets stolen)
- you cannot sell bikes that you don't have in stock

Exercise : Write down the underlying Markov Decision Process

Policies

Definition

A (Markovian) policy is a sequence $\pi = (\pi_t)_{t \in \mathbb{N}^*}$ of mappings

$$\pi_t : \mathcal{S} \to \Delta(\mathcal{A}),$$

where $\Delta(\mathcal{A})$ is the set of probability distributions over the action space.

→ An agent acting under policy π selects at round t the action

$$a_t \sim \pi_t(s_t)$$

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- **Remark** : one could also consider *history-dependent* policies $\pi_t : \mathcal{H}_t \to \Delta(\mathcal{A})$, where the next action is chosen based on

$$h_t = (s_1, a_1, r_1, \dots, s_{t-1}, a_{t-1}, r_{t-1}, s_t)$$

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A policy may be

Deterministic	**Stochastic**
$\pi_t : \mathcal{S} \to \mathcal{A}$	$\pi_t : \mathcal{S} \to \Delta(\mathcal{A})$

- **Terminology** : policy = strategy = decision rule = control

Policies

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A policy may be

Stationary	**Non-stationary**
$\pi = (\pi, \pi, \pi, \dots)$	$\pi = (\pi_1, \pi_2, \dots)$

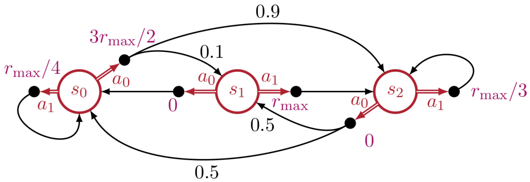
- **Terminology** : policy = strategy = decision rule = control

Policies

Under a stationary (deterministic) policy $\pi : \mathcal{S} \to \mathcal{A}$, the random process $(s_t)_{t\in\mathbb{N}}$ is a Markov chain, with transition probability

$$\mathbb{P}^{\pi}(s_{t+1} = s' | s_t = s) = \mathbb{P}(s_{t+1} = s' | s_t = s, a_t = \pi(s)) = p(s' | s, \pi(s))$$

(can be extended to stochastic policies and continuous spaces)



➜ A MDP is sometimes referred to as a controlled Markov chain

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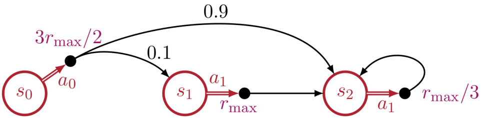

➜ A MDP is sometimes referred to as a controlled Markov chain

Value function

Value of a policy π in a state $s \in \mathcal{S}$

$V^{\pi}(s)$ measures the **expected** cumulative reward obtained by an agent starting from state s and applying policy π.

➜ $\neq$ notions of cumulative reward provide $\neq$ definitions of the value

① **Finite horizon**

Given a known horizon $H \in \mathbb{N}^*$,

$$V^{\pi}(s) = \mathbb{E}^{\pi}\left[\sum_{t=1}^{H} r_t \middle| s_1 = s\right]$$

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starting from state s

➜ **When is it used ?** In the presence of a natural notion of duration of an episode (e.g. maximal number of steps in a game)

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② **Infinite time horizon with a discount parameter**

Given a known discount parameter $\gamma \in (0,1)$,

$$V^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \,\middle|\, s_1 = s\right]$$

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starting from state s

➜ **When is it used ?** To put more weight on short-term reward / when there is a natural notion of discount

Other possible definitions

(not discussed much in this class)

③ **Infinite time horizon with a terminal state**

Given τ the random time at which we first reach a terminal state.

$$V^{\pi}(s) = \mathbb{E}^{\pi}\left[\sum_{t=1}^{\tau} r_t \middle| s_1 = s\right]$$

➜ **When ?** For tasks that have a natural notion of terminal state

④ **Infinite time horizon with average reward**

$$V^{\pi}(s) = \lim_{T\to\infty} \mathbb{E}^{\pi}\left[\frac{1}{T}\sum_{t=1}^{T} r_t \middle| s_1 = s\right]$$

➜ **When ?** The system should be controlled for a very long time

Optimal policy

Given a **value function** (①,②,③ or ④), one can define the following.

Definition

The optimal value in a state s is given by

$$V^\star(s) = \max_\pi V^\pi(s).$$

Theorem [Puterman, 1994]

There exists an optimal policy $\pi^\star$ which satisfies

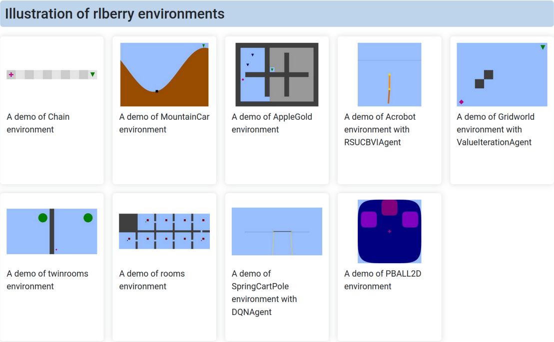
$$\forall s \in \mathcal{S}, \ \pi^\star \in \operatorname*{argmax}_\pi V^\pi(s)$$

Therefore, one can write $V^\star = V^{\pi^\star}$.

➜ as we shall see, one of these optimal policies is deterministic .

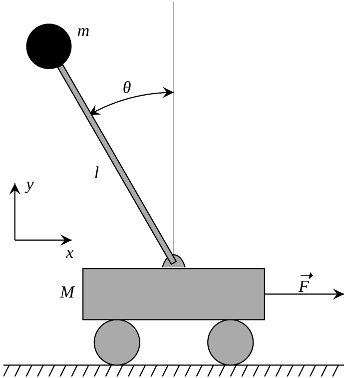
Let’s try some policies on simple environements

To implement environements we will use a structure popularized by the OpenAI Gym library and still used within the rlberry library.



see the introductory notebooks

The Cart Pole environment



Task : maintain the pole as long as possible in a quasi-vertical position, by applying some force on the cart towards the left or right

Back to Retail Store Management

- State : number of bikes in stock on Sunday
  State space : $\mathcal{S} = \{0, \dots, M\}$
- Action : number of bikes ordered at the beginning of the week
  Action space : $\mathcal{A} = \{0, \dots, M\}$
- Reward = balance of the week : if your stock was s_t and you order a_t bikes, in week t you earn

$$r_t = -c_0 \mathbb{1}_{(a_t > 0)} - c \times a_t - h \times s_t + p \times \min(D_t, s_t + a_t, M)$$

- Transition : you end the week with

$$s_{t+1} = \max\left(0, \min(M, s_t + a_t) - D_t\right) \qquad \text{bikes}$$

Goal : From an initial stock s, maximize the sum of discounted rewards

$$V^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \middle| s_1 = s\right]$$

Possible policies

- Uniform policy :

$$\pi(s) \sim \mathcal{U}(\{0, \dots, M - s\})$$

- Constant policy : always buy m_0 bikes

$$\pi(s) = \max(M - s, m_0)$$

- Threshold policy : whenever there are less than m_1 bikes in stock, refill it up to m_2 bikes. Otherwise, do not order.

$$\pi(s) = \mathbb{1}_{(s \leq m_1)}(m_2 - s)$$

Simulations

Let's try out some policies !

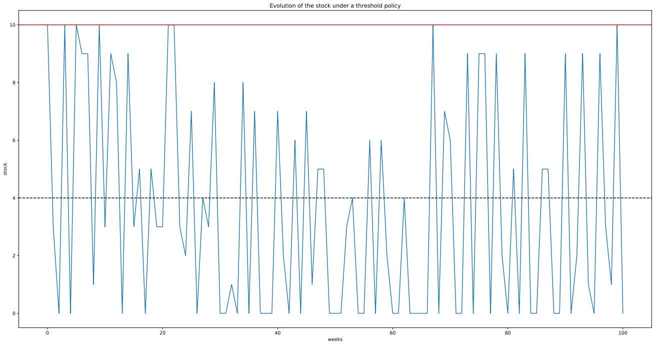


FIGURE – Evolution of the stock s_t under a threshold policy ($m_1 = 4, m_2 = 10$)

Questions

In an known Markov Decision Process

- can we compute an optimal policy ?
  (based on the explicit knowledge of $r(s, a)$ and $p(\cdot|s, a)$)
- ... even with very large (or infinite) state and/or action spaces ?
  (e.g. based on a *simulator* for transitions)

Beyond :

- Can we learn a good policy in an unknown MDP, only by selecting actions and performing transitions ?
- ... and can we do it while maximizing reward ?

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- ... and can we do it while maximizing reward ?

Broad goal of Reinforcement Learning

Learning an optimal **policy** in an unknown (or very large) MDP, by acting (=choosing action) and observing transitions.

Policy evaluation

Given a policy $\pi = (\pi_t)_{t \in \mathbb{N}}$, how can we compute

- Finite horizon MDP :

$$V_h^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=h}^{H} r_t \middle| s_h = s \right]$$

  and in particular $V^\pi(s) = V_1^\pi(s)$

- Discounted MDP :

$$V^\pi(s) = \mathbb{E}^\pi \left[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \middle| s_1 = s \right]$$

Intermezzo : Probability theory

We will need to compute several conditional expectations.

Recall that :

- $\mathbb{E}[X|Y=y]$ is a number :

$$\mathbb{E}[X|Y=y] = \sum_{x\in\mathcal{X}} x\mathbb{P}(X=x|Y=y) \text{ in the discrete case}$$

- $\mathbb{E}[X|Y]$ is a random variable that is $\sigma(Y)$-measurable

$$\mathbb{E}[X|Y] = \sum_{y\in\mathcal{Y}} \mathbb{1}_{(Y=y_i)}\mathbb{E}[X|Y=y_i] \text{ in the discrete case}$$

- more generally $\mathbb{E}[X|\mathcal{F}]$ is random variable that is $\mathcal{F}$-measurable

Useful properties

- Law of total expectation : $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
- $\mathbb{E}[X] = \sum_{y\in\mathcal{Y}} P(Y=y)\mathbb{E}[X|Y=y]$.

Intermezzo : Probability theory

We will need to compute several conditional expectations.

Recall that :

- $\mathbb{E}[X|Y=y]$ is a number :

$$\mathbb{E}[X|Y=y] = \sum_{x \in \mathcal{X}} x\mathbb{P}(X=x|Y=y) \text{ in the discrete case}$$

- $\mathbb{E}[X|Y]$ is a random variable that is $\sigma(Y)$-measurable

$$\mathbb{E}[X|Y] = \sum_{y \in \mathcal{Y}} \mathbb{1}_{(Y=y_i)}\mathbb{E}[X|Y=y_i] \quad \text{in the discrete case}$$

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Useful properties

- Law of total expectation : $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$.
- $\mathbb{E}[X] = \sum_{y \in \mathcal{Y}} P(Y=y)\mathbb{E}[X|Y=y]$.

Bellman equations (finite horizon)

$$V_h^\pi(s) = \mathbb{E}^\pi\left[\sum_{t=h}^{H} r_t \,\middle|\, s_h = s\right]$$

Proposition

The value functions of a deterministic policy π satisfies the following equations : for all $h \in \{1, \dots, H\}$,

$$V_h^\pi(s) \quad = \quad r(s, \pi_h(s)) + \sum_{s' \in \mathcal{S}} p(s'|s, \pi_h(s)) V_{h+1}^\pi(s'),$$

with the convention that $V_{H+1}^\pi(s) = 0$ for all $s \in \mathcal{S}$.

Exercise : Prove it !

Puterman, M. (1994).
Markov Decision Processes. Discrete Stochastic. Dynamic Programming.
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