On the Complexity of A/B Testing

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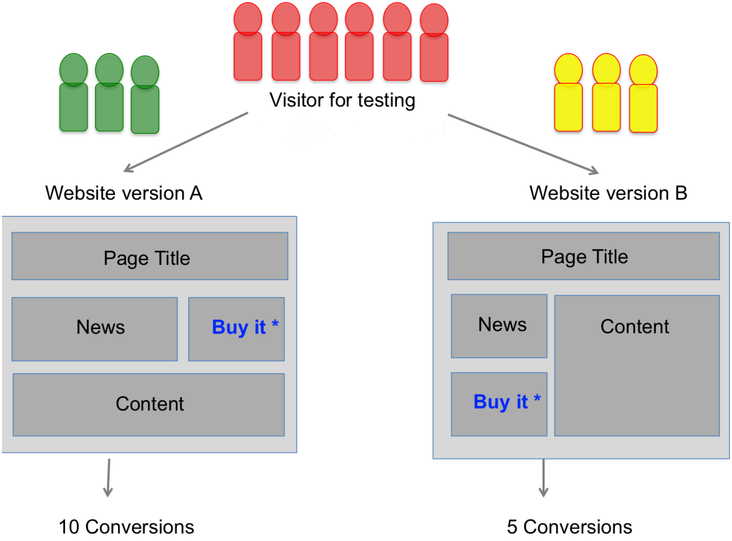
Motivation



Visitor for testing
Website version A
Page Title
News
Buy it *
Content
10 Conversions
Website version B
Page Title
News
Content
Buy it *
5 Conversions

Our goal

Improve performance:

- ➜ fixed number of test users $->$ smaller probability of error
- ➜ fixed probability of error $->$ fewer test users

Tools: sequential allocation and stopping

The model

A two-armed bandit model is

- a set $\nu = (\nu_1, \nu_2)$ of two probability distributions ('arms') with respective means μ_1 and μ_2
- $a^* = \text{argmax}_a \, \mu_a$ is the (unknown) best am

To find the best arm, an agent interacts with the bandit model with

- a *sampling rule* $(A_t)_{t \in \mathbb{N}}$ where $A_t \in \{1, 2\}$ is the arm chosen at time t (based on past observations) $- >$ a sample $Z_t \sim \nu_{A_t}$ is observed
- a *stopping rule* τ indicating when he stops sampling the arms
- a *recommendation rule* $\hat{a}_\tau \in \{1, 2\}$ indicating which arm he thinks is best (at the end of the interaction)

In classical A/B Testing, the sampling rule A_t is uniform on $\{1, 2\}$ and the stopping rule $\tau = t$ is fixed in advance.

Two possible goals

The agent's goal is to design a strategy $\mathcal{A} = ((A_t), \tau, \hat{a}_\tau)$ satisfying

Fixed-budget setting	Fixed-confidence setting
$\tau = t$	$\mathbb{P}_\nu(\hat{a}_\tau \neq a^*) \leq \delta$
$p_t(\nu) := \mathbb{P}_\nu(\hat{a}_t \neq a^*)$ as small as possible	$\mathbb{E}_\nu[\tau]$ as small as possible

An algorithm using uniform sampling is

Fixed-budget setting	Fixed-confidence setting
a classical test of $(\mu_1 > \mu_2)$ against $(\mu_1 < \mu_2)$ based on t samples	a sequential test of $(\mu_1 > \mu_2)$ against $(\mu_1 < \mu_2)$ with probability of error uniformly bounded by δ

[Siegmund 85]: sequential tests can save samples !

The complexities of best-arm identification

Let $\mathcal{M}$ be a class of bandit models. An algorithm $\mathcal{A} = ((A_t), \tau, \hat{a}_\tau)$ is...

Fixed-budget setting	**Fixed-confidence setting**
consistent on $\mathcal{M}$ if $\forall \nu \in \mathcal{M}, p_t(\nu) = \mathbb{P}_\nu(\hat{a}_t \neq a^*) \underset{t\to\infty}{\longrightarrow} 0$	δ-PAC on $\mathcal{M}$ if $\forall \nu \in \mathcal{M},\ \mathbb{P}_\nu(\hat{a}_\tau \neq a^*) \leq \delta$

From the literature

$p_t(\nu) \simeq \exp\left(-\frac{t}{CH(\nu)}\right)$	$\mathbb{E}_\nu[\tau] \simeq C'H'(\nu) \log \frac{1}{\delta}$
[Audibert et al. 10],[Bubeck et al. 11] [Bubeck et al. 13],...	[Mannor Tsitsilis 04],[Even-Dar et al. 06] [Kalanakrishnan et al.12],...

Two complexities

$\kappa_{\text{B}}(\nu) = \inf_{\mathcal{A} \text{ consistent}} \left(\limsup_{t\to\infty} -\frac{1}{t}\log p_t(\nu)\right)^{-1}$	$\kappa_{\text{C}}(\nu) = \inf_{\mathcal{A}\ \delta\text{-PAC}} \limsup_{\delta\to 0} \frac{\mathbb{E}_\nu[\tau]}{\log(1/\delta)}$
for a probability of error $\leq \delta$, budget $t \simeq \kappa_B(\nu) \log \frac{1}{\delta}$	for a probability of error $\leq \delta$ $\mathbb{E}_\nu[\tau] \simeq \kappa_C(\nu) \log \frac{1}{\delta}$

Outline

Changes of distribution

New formulation for a change of distribution

Let ν and ν' be two bandit models. Let N_1 (resp. N_2) denote the total number of draws of arm 1 (resp. arm 2) by algorithm $\mathcal{A}$). For any $A \in \mathcal{F}_\tau$ such that $0 < \mathbb{P}_\nu(A) < 1$

$$\mathbb{E}_\nu[N_1]\text{KL}(\nu_1, \nu_1') + \mathbb{E}_\nu[N_2]\text{KL}(\nu_2, \nu_2') \geq d(\mathbb{P}_\nu(A), \mathbb{P}_{\nu'}(A)),$$

where $d(x,y) := x \log(x/y) + (1-x)\log\big((1-x)/(1-y)\big)$.

General lower bounds

Theorem 1

Let $\mathcal{M}$ be a class of two armed bandit models that are continuously parametrized by their means. Let $\nu = (\nu_1, \nu_2) \in \mathcal{M}$.

Fixed-budget setting	Fixed-confidence setting
any consistent algorithm satisfies $\limsup_{t\to\infty} -\frac{1}{t}\log p_t(\nu) \leq \mathrm{K}^*(\nu_1,\nu_2)$ with $\mathrm{K}^*(\nu_1,\nu_2) = \mathrm{KL}(\nu^*, \nu_1) = \mathrm{KL}(\nu^*, \nu_2)$	any δ-PAC algorithm satisfies $\mathbb{E}_\nu[\tau] \geq \frac{1}{\mathrm{K}_*(\nu_1,\nu_2)}\log\left(\frac{1}{2\delta}\right)$ with $\mathrm{K}_*(\nu_1,\nu_2) = \mathrm{KL}(\nu_1, \nu_*) = \mathrm{KL}(\nu_2, \nu_*)$
Thus, $\kappa_B(\nu) \geq \frac{1}{\mathrm{K}^*(\nu_1,\nu_2)}$	Thus, $\kappa_C(\nu) \geq \frac{1}{\mathrm{K}_*(\nu_1,\nu_2)}$

Fixed-budget setting

For fixed (known) values σ_1, σ_2, we consider Gaussian bandit models

$$\mathcal{M} = \left\{\nu = \left(\mathcal{N}\left(\mu_1, \sigma_1^2\right), \mathcal{N}\left(\mu_2, \sigma_2^2\right)\right) : (\mu_1, \mu_2) \in \mathbb{R}^2, \mu_1 \neq \mu_2\right\}$$

- Theorem 1:

$$\kappa_B(\nu) \geq \frac{2(\sigma_1 + \sigma_2)^2}{(\mu_1 - \mu_2)^2}$$

- A strategy allocating $t_1 = \left\lceil \frac{\sigma_1}{\sigma_1 + \sigma_2} t \right\rceil$ samples to arm 1 and $t_2 = t - t_1$ samples to arm 1, and recommending the empirical best satisfies

$$\liminf_{t\to\infty} -\frac{1}{t} \log p_t(\nu) \geq \frac{(\mu_1 - \mu_2)^2}{2(\sigma_1 + \sigma_2)^2}$$

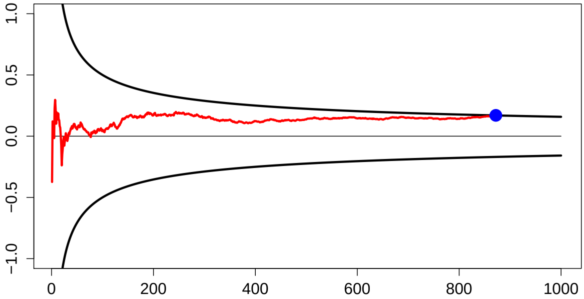
$$\boxed{\kappa_B(\nu) = \frac{2(\sigma_1 + \sigma_2)^2}{(\mu_1 - \mu_2)^2}}$$

Fixed-confidence setting: Algorithm

The α-Elimination algorithm with exploration rate $\beta(t,\delta)$

- ➜ chooses A_t in order to keep a proportion $N_1(t)/t \simeq \alpha$
  i.e. $A_t = 2$ if and only if $\lceil \alpha t \rceil = \lceil \alpha(t+1) \rceil$
- ➜ if $\hat{\mu}_a(t)$ is the empirical mean of rewards obtained from a up to time t, $\sigma_t^2(\alpha) = \sigma_1^2/\lceil \alpha t \rceil + \sigma_2^2/(t - \lceil \alpha t \rceil)$,

$$\tau = \inf\left\{ t \in \mathbb{N} : |\hat{\mu}_1(t) - \hat{\mu}_2(t)| > \sqrt{2\sigma_t^2(\alpha)\beta(t,\delta)} \right\}$$



Fixed-confidence setting: Results

- From Theorem 1:

$$\mathbb{E}_\nu[\tau] \geq \frac{2(\sigma_1+\sigma_2)^2}{(\mu_1-\mu_2)^2}\log\left(\frac{1}{2\delta}\right)$$

- $\frac{\sigma_1}{\sigma_1+\sigma_2}$-Elimination with $\beta(t,\delta) = \log\frac{t}{\delta} + 2\log\log(6t)$ is δ-PAC and

$$\forall \epsilon > 0, \quad \mathbb{E}_\nu[\tau] \leq (1+\epsilon)\frac{2(\sigma_1+\sigma_2)^2}{(\mu_1-\mu_2)^2}\log\left(\frac{1}{2\delta}\right) + \underset{\delta\rightarrow 0}{o_\epsilon}\left(\log\frac{1}{\delta}\right)$$

$$\kappa_C(\nu) = \frac{2(\sigma_1+\sigma_2)^2}{(\mu_1-\mu_2)^2}$$

Gaussian distributions: Conclusions

For any two fixed values of σ_1 and σ_2,

$$\boxed{\kappa_B(\nu) = \kappa_C(\nu) = \frac{2(\sigma_1 + \sigma_2)^2}{(\mu_1 - \mu_2)^2}}$$

If the variances are equal, $\sigma_1 = \sigma_2 = \sigma$,

$$\kappa_B(\nu) = \kappa_C(\nu) = \frac{8\sigma^2}{(\mu_1 - \mu_2)^2}$$

- uniform sampling is optimal only when $\sigma_1 = \sigma_2$
- $1/2$-Elimination is δ-PAC for a smaller exploration rate $\beta(t, \delta) \simeq \log(\log(t)/\delta)$

Lower bounds for Bernoulli bandit models

$$\mathcal{M} = \{\nu = (\mathcal{B}(\mu_1), \mathcal{B}(\mu_2)) : (\mu_1, \mu_2) \in]0;1[^2, \mu_1 \neq \mu_2\},$$

shorthand: $\mathrm{K}(\mu, \mu') = \mathrm{KL}\left(\mathcal{B}(\mu), \mathcal{B}(\mu')\right)$.

Fixed-budget setting	**Fixed-confidence setting**
any consistent algorithm satisfies	any δ-PAC algorithm satisfies
$\limsup_{t\to\infty} -\frac{1}{t}\log p_t(\nu) \leq \mathrm{K}^*(\mu_1, \mu_2)$	$\mathbb{E}_\nu[\tau] \geq \frac{1}{\mathrm{K}_*(\mu_1,\mu_2)} \log\left(\frac{1}{2\delta}\right)$
(Chernoff information)	

$$\boxed{\mathrm{K}^*(\mu_1, \mu_2) > \mathrm{K}_*(\mu_1, \mu_2)}$$

Algorithms using uniform sampling

	For any consistent...	For any δ-PAC...
... algorithm	$p_t(\nu) \gtrsim e^{-\mathrm{K}^*(\mu_1,\mu_2)t}$	$\frac{\mathbb{E}_\nu[\tau]}{\log(1/\delta)} \gtrsim \frac{1}{\mathrm{K}_*(\mu_1,\mu_2)}$
... algorithm using uniform sampling	$p_t(\nu) \gtrsim e^{-\frac{\mathrm{K}(\overline{\mu},\mu_1)+\mathrm{K}(\overline{\mu},\mu_2)}{2}t}$ with $\overline{\mu} = f(\mu_1,\mu_2)$	$\frac{\mathbb{E}_\nu[\tau]}{\log(1/\delta)} \gtrsim \frac{2}{\mathrm{K}(\mu_1,\underline{\mu})+\mathrm{K}(\mu_2,\underline{\mu})}$ with $\underline{\mu} = \frac{\mu_1+\mu_2}{2}$

Remark: Quantities in the same column appear to be close from one another

$\Rightarrow$ **Binary rewards: uniform sampling close to optimal**

Algorithms using uniform sampling

	For any consistent...	For any δ-PAC...
... algorithm	$p_t(\nu) \simeq e^{-\mathrm{K}^*(\mu_1,\mu_2)t}$	$\frac{\mathbb{E}_\nu[\tau]}{\log(1/\delta)} \gtrsim \frac{1}{\mathrm{K}_*(\mu_1,\mu_2)}$
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$\Rightarrow$ **Binary rewards: uniform sampling close to optimal**

Fixed-budget setting

We show that

$$\boxed{\kappa_B(\nu) = \frac{1}{\mathrm{K}^*(\mu_1, \mu_2)}}$$

(matching algorithm not implementable in practice)

The algorithm using uniform sampling and recommending the empirical best arm is preferable (and very close to optimal)

Fixed-confidence setting

δ-PAC algorithms using uniform sampling satisfy

$$\frac{\mathbb{E}_\nu[\tau]}{\log(1/\delta)} \geq \frac{1}{I_*(\nu)} \quad \text{with} \quad I_*(\nu) = \frac{\mathrm{K}\left(\mu_1, \frac{\mu_1+\mu_2}{2}\right) + \mathrm{K}\left(\mu_2, \frac{\mu_1+\mu_2}{2}\right)}{2}.$$

The algorithm using uniform sampling and

$$\tau = \inf\left\{ t \in 2\mathbb{N}^* : |\hat{\mu}_1(t) - \hat{\mu}_2(t)| > \log \frac{\log(t)+1}{\delta} \right\}$$

is δ-PAC but not optimal: $\frac{\mathbb{E}[\tau]}{\log(1/\delta)} \simeq \frac{2}{(\mu_1-\mu_2)^2} > \frac{1}{I_*(\nu)}$.

A better stopping rule NOT based on the difference of empirical means

$$\tau = \inf\left\{ t \in 2\mathbb{N}^* : tI_*(\hat{\mu}_1(t), \hat{\mu}_2(t)) > \log \frac{\log(t)+1}{\delta} \right\}$$

Bernoulli distributions: Conclusion

Regarding the complexities:

- $\kappa_B(\nu) = \frac{1}{\mathrm{K}^*(\mu_1,\mu_2)}$
- $\kappa_C(\nu) \geq \frac{1}{\mathrm{K}_*(\mu_1,\mu_2)} > \frac{1}{\mathrm{K}^*(\mu_1,\mu_2)}$

Thus

$$\boxed{\kappa_C(\nu) > \kappa_B(\nu)}$$

Regarding the algorithms

- There is not much to gain by departing from uniform sampling
- In the fixed-confidence setting, a sequential test based on the difference of the empirical means is no longer optimal

Conclusion

- the complexities $\kappa_B(\nu)$ and $\kappa_C(\nu)$ are not always equal (and feature some different informational quantities)
- for Bernoulli distributions and Gaussian with similar variances, strategies using uniform sampling are (almost) optimal
- strategies using random stopping do not necessarily lead to a saving in terms of the number of sample used

Coming soon:

- Generalization to m best arms identification among K arms