A Spectral Algorithm with Additive Clustering for the Recovery of Overlapping Communities in Networks

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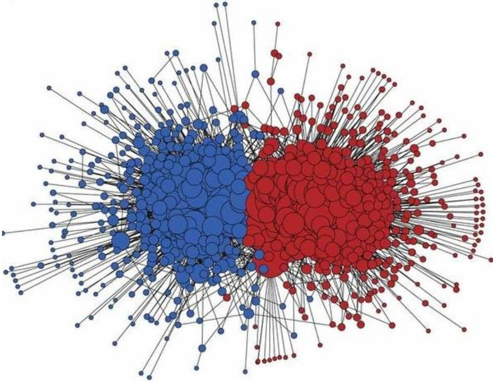
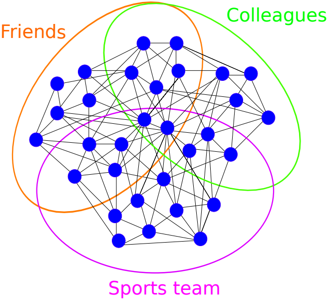
Network partitioning



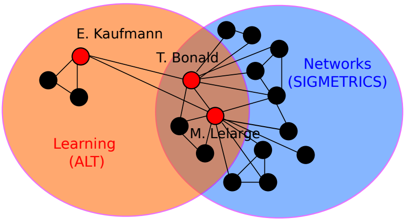
Political blogs network

Overlapping communities : examples

- **Ego-networks**



- **Co-authorship networks**



Our tool : Random graph models

Idea : Assume that the observed graph is drawn from a random graph model that depends on (hidden) communities

- inspires model-based methods for community detection (community detection = estimation problem)
- can be used for evaluation purpose :
  - ➜ try algorithms on simulated data
  - ➜ consistency results : proof that the hidden communities are recovered (if the network is sufficiently large/dense)

Outline

Outline

The Stochastic Block-Model (SBM)

Definition

An undirected, unweighted graph with n nodes is drawn under the random graph model with expected adjacency matrix A if

$$\forall i \leq j, \quad \hat{A}_{i,j} \sim \mathcal{B}(A_{i,j})$$

where $\hat{A}_{i,j}$ is the observed adjacency matrix.

The stochastic block-model with parameter K,Z,B :
[Holland and Leinhard, 1983]

- n nodes, K communities
- a mapping $k : \{1, \ldots, n\} \longrightarrow \{1, \ldots, K\}$
- a connectivity matrix $B \in \mathbb{R}^{K \times K}$

The expected adjacency matrix is

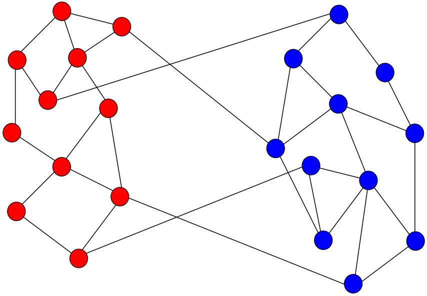
$$A_{i,j} = B_{k(i),k(j)} = (ZBZ^T)_{i,j}$$

for a membership matrix $Z \in \mathbb{R}^{n \times K}$: $Z_{i,l} = \delta_{k(i),l}$.

The Stochastic Block-Model (SBM)

Example : $K = 2$, for $p > q$,

$$B = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$$



The SBMO model

Definition

The Stochastic Block-Model with Overlap (SBMO) has expected adjacency matrix

$$A = ZBZ^T$$

that depends on K, a connectivity matrix $B \in \mathbb{R}^{K \times K}$, and a membership matrix $Z \in \{0,1\}^{n \times K}$.

related models :
MMBS [Airoldi et al. 08], OCCAM [Zhang et al. 14] ...

$Z_i := Z_{i,\cdot} \in \{0,1\}^{1 \times K}$: indicates the communitieS to which node i belongs

Our goal : Given $\hat{A}$ drawn under SBMO, build an estimate $\hat{K}$ of K and $\hat{Z}$ of Z (up to a permutation of its columns).

Identifiability

To perform estimation, the model needs to be identifiable :

$$Z' B' Z'^T = Z B Z^T \quad \Rightarrow \quad \mathrm{Error}(Z', Z) = 0.$$

$$\mathrm{Error}(\hat{Z}, Z) := \frac{1}{nK} \inf_{\sigma \in \mathfrak{S}_K} ||\hat{Z} P_\sigma - Z||_F^2$$

Theorem

The SBMO is identifiable under the following assumptions :

(SBMO1) B is invertible ;

(SBMO2) each community contains at least one pure node :

$$\forall k \in \{1, \dots, K\}, \exists i \in \{1, \dots, n\} : Z_{i,k} = \sum_{\ell=1}^{K} Z_{i,\ell} = 1.$$

Outline

Spectrum of the adjacency matrix under the SBMO

$A = ZBZ^T$ expected adjacency matrix of an identifiable SBMO :

- $Z \in \mathcal{Z} := \{Z \in \{0,1\}^{n \times K}, \forall k \in \{1,\dots,K\} \, \exists i : Z_i = \mathbb{1}_{\{k\}}\}$.
- A is of rank K

$U = [u_1 | \dots | u_K] \in \mathbb{R}^{n \times K}$ a matrix whose columns are normalized eigenvectors associated to the non-zero eigenvalues of A.

Proposition

1. there exists $X \in \mathbb{R}^{K \times K}$: $U = ZX$
2. for all $Z' \in \mathcal{Z}$ and $X' \in \mathbb{R}^{K \times K}$, if $U = Z'X'$, there exists $\sigma \in \mathfrak{S}_k$: $Z = Z'P_\sigma$

($u_1, \dots, u_K$ form a basis of $\mathrm{Im}(A)$ and $\mathrm{Im}(A) \subset \mathrm{Im}(Z)$)

Additive structure in U :

$$\forall i, \;\; U_i = \sum_{k=1}^{K} U_{i_k} \mathbb{1}_{(Z_{i,k}=1)}$$

In the non-overlapping case,

$$U_{i,\cdot} = U_{j,\cdot} \;\Leftrightarrow\; k(i) = k(j)$$

Step 1 : Spectral embedding
Compute $\hat{U} = [\hat{u}_1 | \dots | \hat{u}_K] \in \mathbb{R}^{n \times K}$, matrix of K eigenvectors of $\hat{A}$ associated to largest eigenvalues

$$\text{node } i \;\rightarrow\; \text{vector } \hat{U}_{i,\cdot} \in \mathbb{R}^K$$

Step 2 : Clustering phase
Perform clustering in $\mathbb{R}^K$ on the rows of $\hat{U}$, e.g. K-means clustering

Remarks :

- other possible spectral embeddings (e.g. Laplacian)
- other possible justifications for spectral algorithms

[Von Luxburg 08, Newman 13]

Spectral Algorithm with Additive Clustering

In the general case

$$\forall i,\ U_i = \sum_{k=1}^{K} U_{i_k} \mathbb{1}_{(Z_{i,k}=1)}$$

Step 1 : Spectral embedding
Compute $\hat{U} = [\hat{u}_1 | \dots | \hat{u}_K] \in \mathbb{R}^{n \times K}$, matrix of K eigenvectors of $\hat{A}$ associated to largest eigenvalues
Step 2 : Recovering the additive structure

$$(\mathcal{P}) : \quad (\hat{Z}, \hat{X}) \in \underset{Z' \in \mathcal{Z}, X' \in \mathbb{R}^{K \times K}}{\text{argmin}} ||Z'X' - \hat{U}||_F^2.$$

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In practice : Spectral Algorithm with Additive Clustering computes an (approximate) solution of

$$(\mathcal{P})' : \quad (\hat{Z}, \hat{X}) \in \underset{\substack{Z' \in \{0,1\}^{n \times K} : \forall i, Z'_i \neq 0 \\ X' \in \mathbb{R}^{K \times K}}}{\text{argmin}} \; ||Z'X' - \hat{U}||_F^2.$$

Outline

Goal

- Under which conditions is

$$(\mathcal{P}): \quad (\hat{Z}, \hat{X}) \in \underset{Z' \in \mathcal{Z}, X' \in \mathbb{R}^{K \times K}}{\operatorname{argmin}} ||Z'X' - \hat{U}||_F^2,$$

a good estimation procedure ?

➜ $\hat{U}$ should be close to U

➜ ... and close enough to make $\hat{Z}$ close to Z

Scaling

We analyze the algorithm for a growing network :

$$A = \frac{\alpha_n}{n} Z B Z^T,$$

with α_n a degree parameter, B independent of n, $Z \in \{0,1\}^{n \times K}$:

$$\forall z \in \mathcal{S}, \quad \frac{|\{i : Z_i = z\}|}{n} \to \beta_z > 0.$$

Remark :

$$d_i(n) = \sum_{j=1}^{n} A_{i,j} = \alpha_n \left(\frac{1}{n} Z_i B Z^T \mathbf{1} \right) = O(\alpha_n)$$

Definition : overlap matrix

There exists some matrix $O \in \mathbb{R}^{K \times K}$, called the overlap matrix :

$$\frac{1}{n} Z^T Z \to O.$$

$O_{k,l}$: (limit) proportion of nodes belonging to communities k and l

A precise characterization of the spectrum

The spectrum of A can be related to the spectrum of a $K \times K$ matrix that does not depend on n :

Proposition

Let $\mu \neq 0$. The following statements are equivalent :

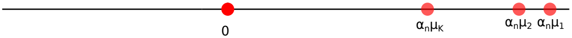
1. x is an eigenvector of $M_0 := O^{1/2} B O^{1/2}$ associated to μ
2. $u = Z O^{-1/2} x$ is an eigenvector of A associated to $\alpha_n \mu$

In particular, the non-zero eigenvalues of A are of order $O(\alpha_n)$.

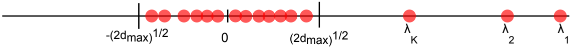
Step 1 : Why is $\hat{U}$ close to U ?

Heuristic :

- Spectrum of A



- Spectrum of $\hat{A} = A +$ perturbation



Extra ingredient : the Davis-Kahan theorem (linear algebra) to prove that the associated eigenvectors are close

Step 1 : Why is $\hat{U}$ close to U ?

$\hat{U} \in \mathbb{R}^{n \times K}$ a matrix that contains normalized eigenvectors of $\hat{A}$ associated with the largest K eigenvalues.

An eigenvectors perturbation result

There exists two constants C, D such that, for all $\delta \in]0,1[$, if

$$d_{\max}(n) \quad \geq \quad C \log\left(n/\delta\right),$$

then with probability larger than $1-\delta$ there exists $\hat{P} \in \mathcal{O}_K(\mathbb{R})$ such that

$$\left|\left|\hat{U} - U\hat{P}\right|\right|_F^2 \leq D\left(\frac{d_{\max}(n)}{\lambda_{\min}(A)^2}\right)\log\left(\frac{4n}{\delta}\right).$$

In the SBMO, $\left\{\begin{array}{lcl} d_{\max}(n) & = & O(\alpha_n) \\ \lambda_{\min}(A) & = & \mu_0\alpha_n \end{array}\right.$: we need $\frac{\alpha_n}{\log(n)} \to \infty$.

Step 2 : Why is $\hat{Z}$ close to Z ?

Let

$$d_0 := \min_{\substack{z \in \{-1,0,1,2\}^{1\times K} \\ z \neq 0}} \left|\left| z O^{-1/2} \right|\right| > 0.$$

Lemma

Let $Z' \in \mathbb{R}^{n\times K}$, $X' \in \mathbb{R}^{K\times K}$ and $\mathcal{N} \subset \{1, \dots, n\}$. Assume that

1. $\forall i \in \mathcal{N}$, $||Z'_i X' - U_i|| \leq \frac{d_0}{4K\sqrt{n}}$
2. there exists $(i_1, \dots, i_K) \in \mathcal{N}^K$ and $(j_1, \dots, j_K) \in \mathcal{N}^K$: $\forall k \in [1, K],\ Z_{i_k} = Z'_{j_k} = \mathbb{1}_{\{k\}}$

Then there exists a permutation matrix P_σ such that

$$\forall i \in \mathcal{N}, Z_i = (Z' P_\sigma)_i.$$

The result

Let ϵ smaller than the smallest proportion of pure nodes and

$$\mathcal{Z}_\epsilon = \left\{ Z' \in \{0,1\}^{n \times K}, \ \forall k \in \{1,\dots,K\}, \frac{|\{i : Z'_i = \mathbb{1}_{\{k\}}\}|}{n} > \epsilon \right\}.$$

Let $\hat{Z}$ be the solution to

$$(\mathcal{P}_\epsilon) : \quad (\hat{Z}, \hat{X}) \in \underset{Z' \in \mathcal{Z}_\epsilon, X' \in \mathbb{R}^{K \times K}}{\text{argmin}} ||Z'X' - \hat{U}||_F^2.$$

Theorem

Assume that $\frac{\alpha_n}{\log n} \to \infty$.
There exists a constant $C_1 > 0$ such that, for n large enough,

$$\mathbb{P}\left(\text{Error}(\hat{Z}, Z) \leq \frac{C_1 K^2}{d_0^2 \mu_0^2} \frac{\log(4n^{1+r})}{\alpha_n} \right) \geq 1 - \frac{1}{n^r}.$$

Remark : Same garantees for the adaptive procedure based on

$$\hat{K} = \left| \left\{ \lambda \in \text{Sp}(\hat{A}) : |\lambda| \geq \sqrt{2(1+\eta)\, \hat{d}_{\max}(n) \log(4n^{1+r})} \right\} \right|$$

Outline

Spectral Algorithm with Additive Clustering (SAAC)

- **Step 1** : spectral embedding based on the adjacency matrix : compute $\hat{U}$, the matrix of K leading eigenvectors of $\hat{A}$
- **Step 2** : compute an approximation of the solution of $(\mathcal{P}')$

$$(\mathcal{P})' : \quad (\hat{Z}, \hat{X}) \in \underset{\substack{Z' \in \{0,1\}^{n \times K} : \forall i, Z'_i \neq 0 \\ X' \in \mathbb{R}^{K \times K}}}{\text{argmin}} ||Z'X' - \hat{U}||_F^2.$$

using alternate minimization.

$$||Z'X' - \hat{U}||_F^2 = \sum_{i=1}^{n} ||Z'_i X' - \hat{U}_i||^2$$

Spectral Algorithm with Additive Clustering (SAAC)

Algorithm 1 Adaptive Combinatorial Spectral Clustering for Overlapping Community Detection

Require: Parameters $\epsilon, r, \eta > 0$. Upper bound on the maximum overlap $O_{\max}$.

Require: $\hat{A}$, the adjacency matrix of the observed graph.

♯ Selection of the eigenvectors

Form $\hat{U}$ a matrix whose columns are $\hat{K}$ eigenvectors of $\hat{A}$ associated to eigenvalues λ satisfying

$$|\lambda| > \sqrt{2(1+\eta)\hat{d}_{\max}(n)\log(4n^{1+r})}$$

♯ Initialization

$\hat{Z} = 0 \in \mathbb{R}^{n \times \hat{K}}$

$\hat{X} \in \mathbb{R}^{\hat{K} \times \hat{K}}$ initialized with k-means++ applied to $\hat{U}$, the first centroid being chosen at random among nodes with degree smaller than the median degree

$Loss = +\infty$

♯ Alternating minimization

while $(Loss - ||\hat{Z}\hat{X} - \hat{U}||_F^2 > \epsilon)$ **do**

  $Loss = ||\hat{Z}\hat{X} - \hat{U}||_F^2$

  Update membership vectors: $\forall i,\ \hat{Z}_{i,\cdot} = \underset{z \in \{0,1\}^{1\times\hat{K}} : 1 \le ||z||_1 \le O_{\max}}{\arg\min} ||\hat{U}_{i,\cdot} - z\hat{X}||$.

  Update centroids: $\hat{X} = (\hat{Z}^T\hat{Z})^{-1}\hat{Z}^T\hat{U}$.

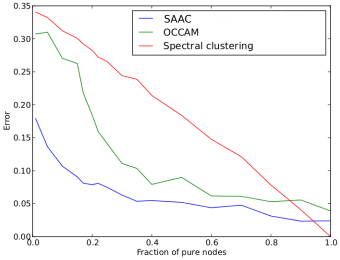
end while

Experiments on simulated data

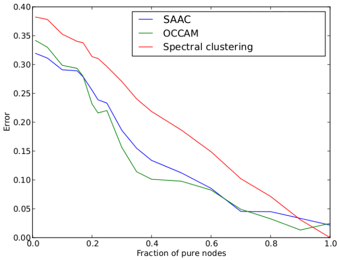
SAAC versus two spectral algorithms :

- Normalized Spectral Clustering (SC)
- the OCCAM spectral algorithm (OCCAM) [Zhang et al. 14]

$n = 500$, $K = 5$, $\alpha_n = (\log n)^{1.5}$, $B = \mathrm{Diag}(5,4,3,3,3)$,
Z : fraction p of pure nodes, $O_{\max} \leq 3$.



under SBMO



under OCCAM

Experiments on real-world networks

Ego-networks from the ego-networks dataset
(SNAP, [Mc Auley, Leskovec 12])

	n	K	c	O_{max}	Error
SC	190 (173)	3.17 (1.07)	1.09 (0.06)	2.17 (0.37)	0.120 (0.083)
OCC.	190 (173)	3.17 (1.07)	1.09 (0.06)	2.17 (0.37)	0.127 (0.102)
SAAC	190 (173)	3.17 (1.07)	1.09 (0.06)	2.17 (0.37)	0.102 (0.049)

TABLE : Spectral algorithms recovering overlapping friend circles in ego-networks from Facebook (average over 6 networks).

Experiments on real-world networks

Co-authorship networks built from DBLP

$$\mathcal{C}_1 = \{\text{NIPS}\},\ \mathcal{C}_2 = \{\text{ICML}\},\ \mathcal{C}_3 = \{\text{COLT}, \text{ALT}\}$$

$$n = 9272,\ \ K = 3,\ \ d_{\text{mean}} = 4.5$$

	c	$\hat{c}$	FP	FN	Error
SC	1.22	1.	0.38	0.39	0.39
OCCAM	1.22	1.02	0.25	0.28	0.27
SAAC	1.22	1.04	0.26	0.28	0.27

$$\mathcal{C}_1 = \{\text{ICML}\},\ \mathcal{C}_2 = \{\text{COLT}, \text{ALT}\}.$$

$$n = 4374,\ \ K = 2,\ \ d_{\text{mean}} = 3.8$$

	c	$\hat{c}$	FP	FN	Error
SC	1.09	1.	0.39	0.55	0.46
OCCAM	1.09	1.00	0.2	0.34	0.26
SAAC	1.09	1.03	0.21	0.31	0.25

Conclusion

SAAC = a spectral algorithm that uses the geometry of the eigenvectors of the adjacency matrix under the SBMO to directly identify overlapping communities

Future work :

- a phase transition in the sparse case ?
- find heuristics for solving $(\mathcal{P}')$ more efficiently
- are other spectral embeddings possible ?
- can the pure nodes assumption be relaxed ?

References

- E. Airoldi, D. Blei, S. Fienberg, and E. Xing. *Mixed Membership Stochastic Block- models*, 2008.
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- M. Newman, *Spectral methods for network community detection and graph partitioning*, 2013
- U. Von Luxburg, *A tutorial on Spectral Clustering*, 2007
- Y. Zhang, E. Levina, J. Zhu, *Detecting Overlapping Communities in Networks with Spectral Methods*, 2014

Identifiability

To perform estimation, the model needs to be identifiable :

$$Z' B' Z'^T = Z B Z^T \quad \Rightarrow \quad \mathrm{MisC}(Z', Z) = 0.$$

- Not always the case ! $Z B Z^T = Z' B' Z'^T = Z'' B'' Z''^T$, with

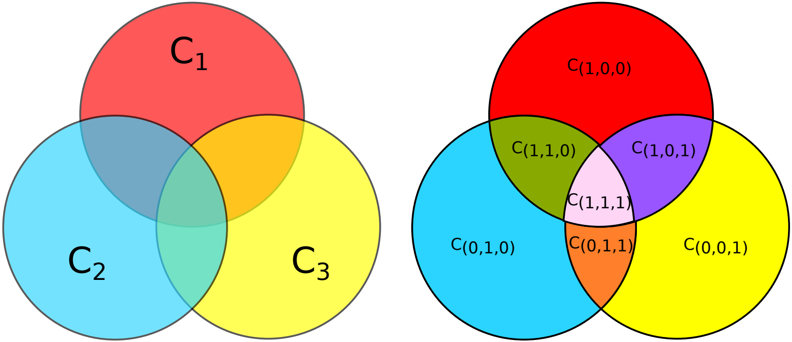
$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad Z = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$B' = \begin{pmatrix} a+b & b & a \\ b & b+c & c \\ a & c & a+c \end{pmatrix} \quad Z' = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$B'' = \begin{pmatrix} a+b-c & b-c & a-c & 0 \\ b-c & b & 0 & 0 \\ a-c & 0 & a & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \quad Z'' = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

SBMO(K,B,Z) can be viewed as a particular case of SBM with

- communities indexed by $\mathcal{S} = \{z \in \{0,1\}^{1\times K} : \exists i : Z_i = z\}$
- $B'_{z,z'} = zBz'^T$



Start by reconstructing the underlying SBM ? Not a good idea.